

Equivariant Perturbation in Gomory and Johnson's Infinite Group Problem

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Abstract

We give an algorithm for testing the extremality of minimal valid functions for Gomory and Johnson's infinite group problem, that are piecewise linear (possibly discontinuous) with rational breakpoints. This is the first set of necessary and sufficient conditions that can be tested algorithmically, for deciding extremality in this important class of minimal valid functions.

1 Introduction

The *infinite group problem* [17, 18] was introduced by Gomory and Johnson as an elegant infinite dimensional generalization of Gomory's corner polyhedron [16]. Both the corner polyhedron and the infinite group problem have played a very important role in the theory of deriving valid cutting planes for integer programming problems. In that regard, these two concepts have had a central role in the foundational aspects of integer programming. Research in this area is still extremely active, with several recent papers discovering very intriguing structures in these problems, and connecting with some deep and beautiful areas of mathematics [1–9, 11–15, 19–21]. Furthermore, there remain many significant open problems which provide fertile grounds for future research. A detailed discussion of the importance of the infinite group problem is beyond the scope of this paper; instead, we refer the interested reader to the recent survey by Conforti, Cornuéjols and Zambelli [10], as well as the research papers cited above.

Gomory's group problem [16] considers an abelian (not necessarily finite) group G , written additively, and studies the set of functions $s : G \rightarrow \mathbb{R}$ satisfying the following constraints:

$$\begin{aligned} \sum_{r \in G} rs(r) &\in f + S & (\text{IR}) \\ s(r) &\in \mathbb{Z}_+ \text{ for all } r \in G \\ s &\text{ has finite support,} \end{aligned}$$

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where f is a given element in G , and S is a subgroup of G ; so $f + S$ is the coset containing the element f .

In particular, we are interested in studying the convex hull $R_f(G, S)$ of all functions satisfying the constraints in (IR). Observe that $R_f(G, S)$ is a convex subset of the (possibly infinite-dimensional) vector space \mathcal{V} of functions $s: G \rightarrow \mathbb{R}$ with finite support. We will be concerned with the so-called *infinite group problem*, where $G = \mathbb{R}$ is taken to be the group of reals under addition, and $S = \mathbb{Z}$ is the subgroup of the integers. A main focus of the research in this area is to give a description of $R_f(\mathbb{R}, \mathbb{Z})$ as the intersection of halfspaces of \mathcal{V} . This makes a very useful connection between $R_f(\mathbb{R}, \mathbb{Z})$ and traditional integer programming, both from a theoretical, as well as, practical point of view. This arises from the fact that important classes of cutting planes for general integer programs can be viewed as finite-dimensional restrictions of the linear inequalities used to describe $R_f(\mathbb{R}, \mathbb{Z})$.

Valid inequalities and valid functions. Any linear inequality in \mathcal{V} is given by a pair (π, α) where π is a function $\pi: G \rightarrow \mathbb{R}$ (not necessarily of finite support) and $\alpha \in \mathbb{R}$. The linear inequality is then given by $\sum_{r \in G} \pi(r)s(r) \geq \alpha$; the left-hand side is a finite sum because s has finite support. Such an inequality is called a *valid inequality* for $R_f(G, S)$ if $\sum_{r \in G} \pi(r)s(r) \geq \alpha$ for all $s \in R_f(G, S)$.

For historical and technical reasons, we concentrate on those valid inequalities for which $\pi \geq 0$. This implies that we can choose, after a scaling, $\alpha = 1$. Thus, we only focus on valid inequalities of the form $\sum_{r \in G} \pi(r)s(r) \geq 1$ with $\pi \geq 0$. Such functions π will be termed *valid functions* for $R_f(G, S)$. As pointed out in [10], the non-negativity assumption in the definition of a valid function might seem artificial at first. Although there might exist valid inequalities $\sum_{r \in \mathbb{R}} \pi(r)s(r) \geq \alpha$ for $R_f(\mathbb{R}, \mathbb{Z})$ such that $\pi(r) < 0$ for some $r \in \mathbb{R}$, it can be shown that π must be non-negative over all *rational* $r \in \mathbb{R}$. Since data in integer programs are usually rational, it is natural to focus on non-negative valid functions.

Minimal and extreme functions. Gomory and Johnson [17, 18] defined a hierarchy on the set of valid functions, capturing the strength of the corresponding valid inequalities, which we summarize now. A valid function π for $R_f(G, S)$ is said to be *minimal* for $R_f(G, S)$ if there is no valid function $\pi' \neq \pi$ such that $\pi'(r) \leq \pi(r)$ for all $r \in G$. It is known that for every valid function π for $R_f(G, S)$, there exists a minimal valid function π' such that $\pi' \leq \pi$ (see [8] for a proof in the case when $G = \mathbb{R}^k$, $S = \mathbb{Z}^k$). Since $s \in R_f(G, S)$ are always nonnegative functions, minimal functions clearly dominate valid functions that are not minimal, making the latter redundant in the description of $R_f(G, S)$. A stronger notion is that of an *extreme function*. A valid function π is *extreme* for $R_f(G, S)$ if it cannot be written as a convex combination of two other valid functions for $R_f(G, S)$, i.e., $\pi = \frac{1}{2}\pi_1 + \frac{1}{2}\pi_2$ implies $\pi = \pi_1 = \pi_2$. It is easy to verify that extreme functions are minimal. Minimal functions for $R_f(G, S)$ were well characterized by Gomory for finite groups G in [16], and later for $R_f(\mathbb{R}, \mathbb{Z})$ by Gomory and Johnson [17]. We state these results in a unified notation in the following theorem.

A function $\pi: G \rightarrow \mathbb{R}$ is *subadditive* if $\pi(x + y) \leq \pi(x) + \pi(y)$ for all $x, y \in G$. We say that π is *symmetric* (or *satisfies the symmetry condition*) if $\pi(x) + \pi(f - x) = 1$ for all $x \in G$.

Theorem 1.1 (Gomory and Johnson [17]). *Let $\pi: G \rightarrow \mathbb{R}$ be a non-negative function. Then π is a minimal valid function for $R_f(G, S)$ if and only if $\pi(r) = 0$ for all $r \in S$, π is subadditive, and π satisfies the symmetry condition. The first two conditions imply that π is constant over any coset of S .*

Remark 1.2. Note that this implies that one can view a minimal valid function π as a function from G/S to \mathbb{R} , and thus studying $R_f(G, S)$ is the same as studying $R_f(G/S, 0)$. However, we avoid this viewpoint in this paper.

A tight characterization of extreme functions for $R_f(\mathbb{R}, \mathbb{Z})$ has eluded researchers for the past four decades now. In this paper, we give an algorithm for deciding the extremality of piecewise linear functions with rational breakpoints. This algorithm is then used to prove a simple necessary and sufficient condition for the extremality of *continuous* piecewise linear minimal functions. To the best of our knowledge, these two results are the first of their kind; in comparison, the majority of results on extremality in the literature are very specific sufficient conditions for guaranteeing extremality [8, 12–15, 19]. Moreover, some of the more general sufficient conditions (see, for example, Theorem 6 in [15]) are not algorithmic in the sense that given a particular valid function, it is not possible to test the sufficient condition in a finite number of computations. We give more details below.

Overview of our main results. For the infinite group problem $R_f(\mathbb{R}, \mathbb{Z})$, by Theorem 1.1 a minimal valid function $\pi: \mathbb{R} \rightarrow \mathbb{R}_+$ is periodic with period 1. We consider piecewise linear functions, possibly discontinuous, whose breakpoints in the fundamental domain $[0, 1]$ form a finite set B .

Our first main result is an algorithm for deciding if a given piecewise linear function is extreme or not. The proof of this theorem appears in section 3.3.

Theorem 1.3. *Consider the following problem.*

Given a minimal valid function π for $R_f(\mathbb{R}, \mathbb{Z})$ that is piecewise linear with a set of rational breakpoints with the least common denominator q , decide if π is extreme or not.

There exists an algorithm for this problem that takes a number of elementary operations over the reals that is bounded by a polynomial in q .

If we start with any piecewise linear valid function π , the first step is to determine if π is minimal. A minimality test for *continuous* piecewise linear functions was given by Gomory and Johnson (see Theorem 7 in [17]). In section 2.3 we present a minimality test that works for discontinuous functions too.

Next we investigate the precise relation between continuous extreme functions for the infinite group problem and certain finite group problems, i.e., $R_f(G, \mathbb{Z})$ where G is a discrete subgroup of \mathbb{R} . A first result in this direction appeared in [15]; we state it in our notation.

Theorem 1.4 (Theorem 6 in [15]). *Let π be a piecewise linear minimal valid function for $R_f(\mathbb{R}, \mathbb{Z})$ with set B of rational breakpoints with the least common denominator q . Let $G_{2^n}, n \in \mathbb{N}$ denote the subgroups $\frac{1}{2^n q} \mathbb{Z}$. Then π is extreme if and only if the restriction $\pi|_{G_{2^n}}$ is extreme for $R_f(G_{2^n}, \mathbb{Z})$ for all $n \in \mathbb{N}$.*

Clearly the above condition cannot be checked in a finite number of steps and hence cannot be converted into an algorithm, because it potentially needs to test infinitely many finite group problems. In contrast, we prove the following result in section 3.3.

Theorem 1.5. *Let π be a continuous piecewise linear minimal valid function for $R_f(\mathbb{R}, \mathbb{Z})$ with set B of rational breakpoints with the least common denominator q . Let $\hat{G} = \frac{1}{4q}\mathbb{Z}$. Then π is extreme for $R_f(\mathbb{R}, \mathbb{Z})$ if and only if $\pi|_{\hat{G}}$ is extreme for $R_f(\hat{G}, \mathbb{Z})$.*

Gomory gives a characterization of extreme valid functions for finite group problems via a linear program. This provides an alternative algorithm for testing extremality of *continuous* piecewise linear function with rational breakpoints, under the light of Theorem 1.5. Of course, Theorem 1.3 is more general as it provides an algorithm for discontinuous functions also.

Techniques of this paper. The standard technique for showing extremality is to suppose that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, where π^1, π^2 are other (minimal) valid functions. All subadditivity relations that are tight for π are also tight for π^1, π^2 . Then one shows that actually $\pi = \pi^1 = \pi^2$ holds. The main tool used in the literature for showing this is the so-called Interval Lemma introduced by Gomory and Johnson in [19], which we state here for a more coherent discussion of the new ideas in this paper.

Lemma 1.6 (Interval Lemma [7, 19]). *Let $\theta: \mathbb{R} \rightarrow \mathbb{R}$ be a function bounded on every bounded interval. Given real numbers $u_1 < u_2$ and $v_1 < v_2$, let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$. If $\theta(u) + \theta(v) = \theta(u + v)$ for every $u \in U$ and $v \in V$, then there exists $c \in \mathbb{R}$ such that*

$$\begin{aligned} \theta(u) &= \theta(u_1) + c(u - u_1) && \text{for every } u \in U, \\ \theta(v) &= \theta(v_1) + c(v - v_1) && \text{for every } v \in V, \\ \theta(w) &= \theta(u_1 + v_1) + c(w - u_1 - v_1) && \text{for every } w \in U + V. \end{aligned}$$

Remark 1.7. We remark that the Interval Lemma is a lemma of *real analysis* (the theory of functional equations); the hypothesis that the function θ is bounded on every bounded interval is one of several possible hypotheses to rule out certain pathological functions.

Every proof of extremality in the existing literature employs the Interval Lemma on disjoint intervals that satisfy certain additivity conditions to deduce affine linearity properties that π^1 and π^2 share with π . This is followed by a *linear algebra* argument (explicit in Gomory–Johnson’s proof of the 2-Slope Theorem, but implicit in many other proofs) to establish uniqueness of π , and thus its extremality.

Surprisingly, the *arithmetic* (number-theoretic) aspects of the problem seem to have been largely overlooked, even though they are at the core of the theory of the closely related *finite* group problem. This aspect turns out to be the key for completing the algorithmic classification of extreme piecewise linear functions.

To capture the relevant arithmetics of the problem, we study finite sets of additivity relations of the form $\pi(t_i) + \pi(y) = \pi(t_i + y)$ and $\pi(x) + \pi(r_i - x) = \pi(r_i)$, where the points t_i and r_i are certain breakpoints of the function π . They give rise to a useful abstraction, the *reflection group* Γ generated by the reflections $\rho_{r_i}: x \mapsto r_i - x$ and translations $\tau_{t_i}: y \mapsto t_i + y$.

We then study the natural action of the reflection group Γ on the set of open intervals delimited by the elements of $G = \frac{1}{q}\mathbb{Z}$. Roughly speaking, the action of Γ transfers the affine linearity established by the Interval Lemma on some interval I to the orbit $\Gamma(I)$. Actually, this transfer is more delicate, and we have to combine this arithmetic consideration with a discussion of reachability. In the end, the transfer happens within each connected component of a certain graph.

When the Interval Lemma and this transfer technique establish affine linearity of π^1, π^2 on all intervals where π is affinely linear, we can proceed with linear algebra techniques to decide extremality of π . Otherwise, we show that there is a way to perturb π slightly to construct distinct minimal valid functions $\pi^1 = \pi + \bar{\pi}$ and $\pi^2 = \pi - \bar{\pi}$. Here the reflection group Γ gives a blueprint for this perturbation. We can choose the perturbation function $\bar{\pi}$ to be any sufficiently small, Γ -equivariant function, again modified by restriction to a certain connected component.

An interesting irrational function and a complexity conjecture. We now discuss the complexity of the algorithm of Theorem 1.3. For the purpose of this discussion, let us restrict ourselves to a version of the problem where all input data are rational. It is an open question whether the pseudo-polynomial complexity of our algorithm is best possible, or whether there exists a polynomial-time algorithm, or even a strongly polynomial-time algorithm, whose running time would only depend on the number of breakpoints but not on the sizes of the denominators. We conjecture that the problem (in a suitable version with all rational input data) is NP-hard in the weak sense.

We believe that the problem is intrinsically arithmetic, so that an algorithm that is oblivious to the sizes of the denominators is not possible. To substantiate this, we construct a certain extreme function with irrational breakpoints (section 4). Any nearby function with rational breakpoints that uses the same construction turns out to not be extreme.

The proof of extremality of this function requires another technique unrelated to the Interval Lemma. Here a reflection group Γ arises under which every point x has an orbit $\Gamma(x)$ that is dense in \mathbb{R} . Roughly speaking (ignoring the reachability issues, which our proof has to discuss), there exists no non-trivial *continuous* Γ -equivariant perturbation. Thus the function is extreme.

2 Preliminaries

2.1 Discontinuous piecewise linear functions

We first give a definition of piecewise linear minimal functions that allows for discontinuous functions; see also Figure 1. Let B be a set of breakpoints $0 = x_0 < x_1 < \dots < x_n = 1$. Let the set of 0-faces be the collection of singletons, $\mathcal{I}_{B,0} = \{\{x_0\}, \{x_1\}, \dots, \{x_n\}\}$, and the set of 1-faces be the collection $\mathcal{I}_{B,1}$ of intervals $[x_i, x_{i+1}]$ for $i = 0, \dots, n-1$. For each 0-face $I \in \mathcal{I}_{B,0}$, there is a constant function $\pi_I(x) = b_I$; for each 1-face $I \in \mathcal{I}_{B,1}$, there is an affine linear function $\pi_I(x) = m_I x + b_I$, defined for all $x \in \mathbb{R}$. A minimal function $\pi: \mathbb{R} \rightarrow \mathbb{R}_+$ is called *piecewise linear* if it is given by $\pi(x) = \pi_I(x)$ where $x \in \text{rel int}(I)$ for some $I \in \mathcal{I}_{B,0} \cup \mathcal{I}_{B,1}$ on its fundamental domain $[0, 1]$.

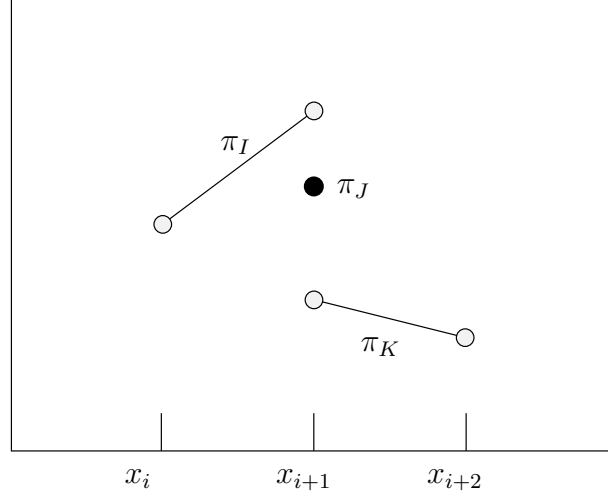


Figure 1: A discontinuous piecewise linear function π with breakpoints $B = \{x_0, \dots, x_n\}$ with $0 = x_0 < x_1 < \dots < x_n = 1$. This figure shows a piecewise linear function π on (x_i, x_{i+2}) where $I = [x_i, x_{i+1}]$, $J = \{x_{i+1}\}$, $K = [x_{i+1}, x_{i+2}]$.

We will use \oplus and \ominus to denote addition and subtraction modulo 1, respectively. We first show that f must be a breakpoint of any minimal valid function.

Lemma 2.1. *If π is a minimal function, then $f \in B$.*

Proof. First, suppose that there exists an $x \in [0, 1]$ such that $\pi(x) > 1$. Since π is symmetric, $\pi(x) + \pi(f \ominus x) = 1$, which implies that $\pi(f \ominus x) < 0$, which is a contradiction since we assumed that $\pi \geq 0$. Therefore $0 \leq \pi \leq 1$.

Now, suppose that $f \notin B$, i.e., $f \in I$ for some $I \in \mathcal{I}_{B,1}$. Symmetry and the condition that $\pi(0) = 0$ imply that $\pi(f) = 1$. Since $\pi \leq 1$ and π is affine in I , it follows that $\pi \equiv 1$ on I . The set $f \ominus I$ contains the origin and by symmetry, $\pi \equiv 0$ on $f \ominus I$. Let J be the largest interval containing the origin such that $\pi \equiv 0$ on J and let $\bar{x} = \sup\{x \in J \mid x > 0\}$. Since J is the largest such interval, and $J \neq [0, 1]$, for every small $\epsilon > 0$, there exists a point $y \geq \bar{x}$ such that $\epsilon, y \ominus \epsilon \in J$ and $\pi(y) > 0$. But then $\pi(\epsilon) + \pi(y \ominus \epsilon) = 0 < \pi(y)$, which violates subadditivity, and therefore is a contradiction. \square

2.2 A polyhedral complex

The following notation will be used in the rest of the paper. The function $\Delta\pi$ measures the slack in the subadditivity constraints:

$$\Delta\pi(x, y) = \pi(x) + \pi(y) - \pi(x \oplus y).$$

Let \mathcal{P}_B be the polyhedral complex with faces

$$F = \{ (x, y) \in \mathbb{R}^2 \mid x \in I, y \in J, x \oplus y \in K \}$$

where $I, J, K \in \mathcal{I}_{B,0} \cup \mathcal{I}_{B,1}$; see Figure 2.

Observe that $\Delta\pi|_{\text{rel int}(F)}$ is affine; if we introduce the function $\Delta\pi_F(x, y) = \pi_I(x) + \pi_J(y) - \pi_K(x \oplus y)$ for all $x, y \in \mathbb{R}$, then $\Delta\pi(x, y) = \Delta\pi_F(x, y)$ for all $(x, y) \in \text{rel int}(F)$. We will use $\text{vert}(F)$ to denote the set of vertices of the face F .

Remark 2.2. This polyhedral complex was studied by Gomory and Johnson in [19] for a different purpose (the so-called merit index).

2.3 Finite test for minimality of piecewise linear functions

In this subsection, we show that there is an easy test to see if a piecewise linear function is minimal. A minimality test for *continuous* piecewise linear functions was given by Gomory and Johnson (see Theorem 7 in [17]). The minimality test we present in this section works for discontinuous functions too.

Lemma 2.3. *Suppose that $\pi(f) = 1$.*

1. *π is subadditive if and only if for all $F \in \mathcal{P}_B$, $\Delta\pi_F(u, v) \geq 0$ for all $(u, v) \in \text{vert}(F)$,*
2. *π is symmetric if and only if for all $F \in \mathcal{P}_B$ with $F \subseteq \{(x, y) \mid x \oplus y = f\}$, we have that $\Delta\pi_F(u, v) = 0$ for all $(u, v) \in \text{vert}(F)$.*

Proof. First we show that the stated conditions are necessary. Since $\pi(f) = 1$, whenever $\Delta\pi(x, f \ominus x) = 0$, i.e., $\pi(x) + \pi(x \ominus f) = \pi(f) = 1$, we have that π is symmetric at x .

Let $F \in \mathcal{P}_B$ and let $(u, v) \in \text{vert}(F)$. Then

$$\Delta\pi_F(u, v) = \lim_{\substack{(x, y) \rightarrow (u, v) \\ (x, y) \in \text{rel int}(F)}} \Delta\pi(x, y). \quad (1)$$

If π is subadditive, then $\Delta\pi(x, y) \geq 0$ for all $x, y \in [0, 1]$, and therefore the limit above is also non-negative. Suppose $F \subseteq \{(x, y) \mid x \oplus y = f\}$. If π is symmetric, then $\Delta\pi(x, y) = 0$ whenever $x \oplus y = f$ and $x, y \in [0, 1]$. Therefore, the above limit is in fact a limit of zeros, and $\Delta\pi_F(u, v) = 0$.

We now show that the stated conditions are sufficient.

For subadditivity, we need to show that $\Delta\pi(x, y) \geq 0$ for all $x, y \in [0, 1]$. Let $x, y \in [0, 1]$, then $(x, y) \in \text{rel int}(F)$ for some unique $F \in \mathcal{P}_B$ and $\Delta\pi(x, y) = \Delta\pi_F(x, y)$. Since $\Delta\pi_F(u, v) \geq 0$ for all $(u, v) \in \text{vert}(F)$, by convexity ($\Delta\pi_F$ is affine), $\Delta\pi_F(x, y) \geq 0$. Therefore π is subadditive.

Similarly, to show symmetry, we need to show that $\Delta\pi(x, y) = 0$ for all $x, y \in [0, 1]$ such that $x \oplus y = f$. Let $x, y \in [0, 1]$ such that $x \oplus y = f$. Since $f \in B$ by Lemma 2.1, $(x, y) \in \text{rel int}(F)$ for some $F \in \mathcal{P}_B$ for $F \subseteq \{(x, y) \mid x \oplus y = f\}$. Since $\Delta\pi_F(u, v) = 0$ for all $(u, v) \in \text{vert}(F)$, and $\Delta\pi_F$ is affine, it follows that $\Delta\pi(x, y) = \Delta\pi_F(x, y) = 0$. \square

Remark 2.4. This lemma implies that there are, for $n = |B|$, $O(n^2)$ pairs that need to be checked in order to check subadditivity, and only $O(n)$ points that need to be evaluated to check for symmetry. These follow from the fact that there are $O(n)$ hyperplanes in the polyhedral complex. Any 0-dimensional face is at the intersection of at least two hyperplanes, therefore, at most $\binom{O(n)}{2}$ possibilities. And any 0-dimensional face corresponding to a symmetry condition is at the intersection of the hyperplane $x + y = f$ and any other one, therefore, at most $O(n)$ such points.

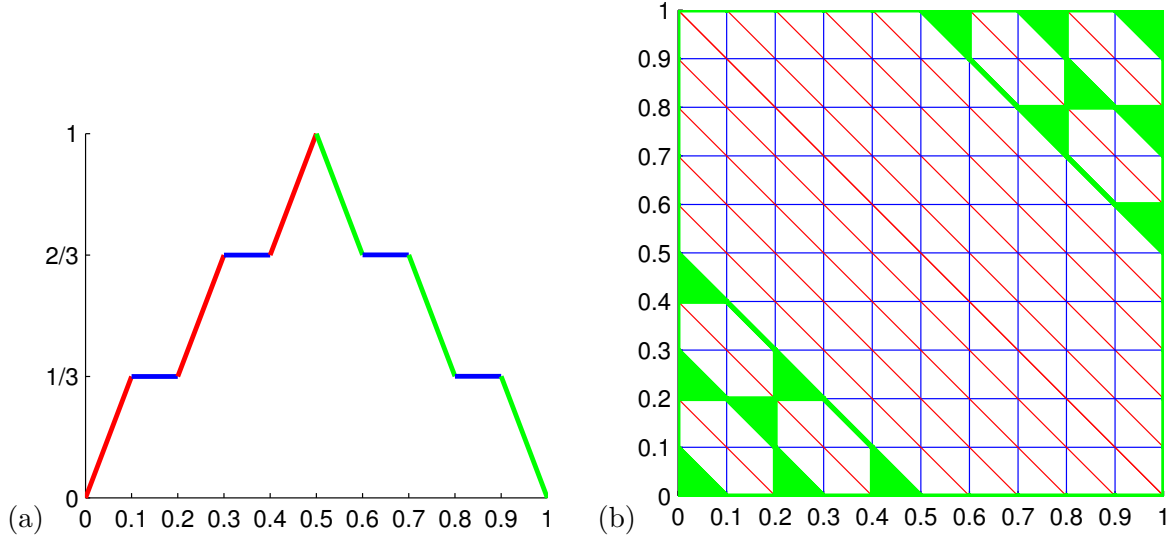


Figure 2: A minimal continuous piecewise linear function π and its polyhedral complex \mathcal{P}_B . (a) The function π with $f = 0.5$ and breakpoints in B . Here $G = \langle B \rangle_{\mathbb{Z}} = \frac{1}{10}\mathbb{Z}$. The function is not extreme, but is affine imposing in $\mathcal{I}_{G,1}$ with three slopes: a positive slope, a negative slope, and zero slope. The intervals of positive slope are all connected in \mathcal{G} , as well as the intervals of negative slope. But not all the intervals of zero slope are connected in \mathcal{G} . The zero slope intervals on the left side and the zero slope intervals on the right side form two separate connected components of the graph. (b) Its polyhedral complex \mathcal{P}_B . The polyhedral complex is shaded bright green wherever $\Delta\pi(x, y) = 0$.

Theorem 2.5 (Minimality test). *A piecewise linear function $\pi: \mathbb{R} \rightarrow \mathbb{R}$ with breakpoints B and period 1 is minimal and only if*

1. $\pi(0) = 0, \pi(f) = 1$,
2. for all $F \in \mathcal{P}_B$, $\Delta\pi_F(u, v) \geq 0$ for all $(u, v) \in \text{vert}(F)$,
3. for all $F \in \mathcal{P}_B$ with $F \subseteq \{(x, y) \mid x \oplus y = f\}$, we have that $\Delta\pi_F(u, v) = 0$ for all $(u, v) \in \text{vert}(F)$.

2.4 Continuity results

We will need the following lemma and theorem on continuity. Although similar results appear in [18], we provide proofs of these facts to keep this paper more self-contained.

Lemma 2.6. *If $\theta: \mathbb{R} \rightarrow \mathbb{R}$ is a subadditive function and $\limsup_{h \rightarrow 0} |\frac{\theta(h)}{h}| = L < \infty$, then $\theta(h)$ is Lipschitz continuous with Lipschitz constant L .*

Proof. Fix any $\delta > 0$. Since $\limsup_{h \rightarrow 0} |\frac{\theta(h)}{h}| = L$, there exists $\epsilon > 0$ such that for any $x, y \in \mathbb{R}$ satisfying $|x - y| < \epsilon$, $|\frac{\theta(x-y)}{|x-y|}| < L + \delta$. By subadditivity, $|\theta(x - y)| \geq |\theta(x) - \theta(y)|$ and so $|\frac{\theta(x) - \theta(y)}{|x - y|}| < L + \delta$ for all $x, y \in \mathbb{R}$ satisfying $|x - y| < \epsilon$. This immediately implies

that for all $x, y \in \mathbb{R}$, $\frac{|\theta(x) - \theta(y)|}{|x - y|} < L + \delta$, by simply breaking the interval $[x, y]$ into equal subintervals of size at most ϵ . Since the choice of δ was arbitrary, this shows that for every $\delta > 0$, $\frac{|\theta(x) - \theta(y)|}{|x - y|} < L + \delta$ and therefore, $\frac{|\theta(x) - \theta(y)|}{|x - y|} \leq L$. Therefore, θ is Lipschitz continuous with Lipschitz constant L . \square

Theorem 2.7. *If $\pi: \mathbb{R} \rightarrow \mathbb{R}$ is a minimal valid function, and $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, where π^1, π^2 are valid functions, then π^1, π^2 are both minimal. Moreover, if $\limsup_{h \rightarrow 0} |\frac{\pi(h)}{h}| < \infty$, then this condition also holds for π^1 and π^2 . This implies that π, π^1 and π^2 are all Lipschitz continuous.*

Proof. The minimality of π^1, π^2 is clear. Since we assume $\pi^1, \pi^2 \geq 0$, $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$ implies that $\pi^i \leq 2\pi$ for $i = 1, 2$. Therefore if $\limsup_{h \rightarrow 0} |\frac{\pi(h)}{h}| = L < \infty$, then $\limsup_{h \rightarrow 0} |\frac{\pi^i(h)}{h}| \leq 2L < \infty$ for $i = 1, 2$. Applying Lemma 2.6, we get Lipschitz continuity for all three functions. \square

2.5 Finitely generated reflection groups from additivity relations

We need to study the pairs (u, v) where the subadditivity condition is satisfied at equality, i.e., $\pi(u) + \pi(v) = \pi(u \oplus v)$, or, equivalently, $\Delta\pi(u, v) = 0$. These additivity relations also hold for subadditive functions π^1, π^2 if $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$. By introducing the difference function (perturbation) $\bar{\pi} = \pi^1 - \pi$, we can write $\pi^1 = \pi + \bar{\pi}$ and $\pi^2 = \pi - \bar{\pi}$. Then the same additivity relations also hold for $\bar{\pi}$.

In this section, we develop a way to use these additivity relations that complements the standard use of the Interval Lemma, where one looks for closed non-degenerate intervals U, V, W such that $\pi(u) + \pi(v) = \pi(u + v)$ for all $u \in U$ and $v \in V$ with $u + v \in W$.

Here we consider such relations when one of U, V , or W is a single point, instead of a non-degenerate interval. So let us assume that $\pi: \mathbb{R} \rightarrow \mathbb{R}$ satisfies finitely many classes of relations of the type

$$\pi(t_i) + \pi(y) = \pi(t_i + y), \quad i = 1, \dots, m \quad (2a)$$

for all y in some interval Y_i and

$$\pi(x) + \pi(r_i - x) = \pi(r_i), \quad i = 1, \dots, n \quad (2b)$$

for all x in some interval X_i , where t_1, \dots, t_m and r_1, \dots, r_n are finitely many points in \mathbb{R} .

Under some conditions we will be able to construct a perturbation $\bar{\pi}$ to create valid functions π^1 and π^2 from π . Our strategy is to first construct a function ψ that satisfies more conditions, namely equation (2a) for all $y \in \mathbb{R}$ and equation (2b) for all $x \in \mathbb{R}$.

For this construction we can use methods of group theory. We consider a subgroup of the group $\text{Aff}(\mathbb{R})$ of invertible affine linear transformations of \mathbb{R} as follows.

Definition 2.8. For a point $r \in \mathbb{R}$, define the *reflection* $\rho_r: \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto r - x$. For a vector $t \in \mathbb{R}$, define the *translation* $\tau_t: \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x + t$.

Given a finite number of points r_1, \dots, r_n and a finite number of vectors t_1, \dots, t_m , we will define the subgroup

$$\Gamma = \langle \rho_{r_1}, \dots, \rho_{r_n}, \tau_{t_1}, \dots, \tau_{t_m} \rangle.$$

Let $r, s, w, t \in \mathbb{R}$. Each reflection is an involution: $\rho_r \circ \rho_r = id$, two reflections give one translation: $\rho_r \circ \rho_s = \tau_{r-s}$. Thus, if we assign a *character* $\chi(\rho_r) = -1$ to every reflection and $\chi(\tau_t) = +1$ to every translation, then this extends to a *group character* of Γ , that is, a group homomorphism $\chi: \Gamma \rightarrow \mathbb{C}^\times$.

On the other hand, not all pairs of reflections need to be considered: $\rho_s \circ \rho_w = (\rho_s \circ \rho_r) \circ (\rho_r \circ \rho_w) = (\rho_r \circ \rho_s)^{-1} \circ (\rho_r \circ \rho_w)$. Thus the subgroup $T = \ker \chi$ of translations in Γ is generated as follows.

$$T = \langle \tau_{r_2-r_1}, \dots, \tau_{r_n-r_1}, \tau_{t_1}, \dots, \tau_{t_m} \rangle.$$

It is *normal* in Γ , as it is stable by conjugation by any reflection: $\rho_r \circ \tau_t \circ \rho_r^{-1} = \tau_{-t}$. If $\gamma \in \Gamma$ is not a translation, i.e., $\chi(\gamma) = -1$, then it is generated by an odd number of reflections, and thus can be written as $\gamma = \tau \rho_{r_1}$ with $\tau \in T$. Thus $\Gamma/T = \langle \rho_{r_1} \rangle$ is of order 2. In short, we have the following lemma.

Lemma 2.9. *The group Γ is the semidirect product $T \rtimes \langle \rho_{r_1} \rangle$, where the (normal) subgroup of translations can be written as*

$$T = \{ \tau_t \mid t \in \Lambda \},$$

where Λ is the additive subgroup

$$\Lambda = \langle r_2 - r_1, \dots, r_n - r_1, t_1, \dots, t_m \rangle_{\mathbb{Z}} \subseteq \mathbb{R}.$$

Next we show how to construct functions that are Γ -equivariant, i.e., invariant under T and odd w.r.t. all reflections $\rho_{r_1}, \dots, \rho_{r_n}$.

Lemma 2.10 (Construction of Γ -equivariant functions). *Suppose that the additive subgroup Λ defined above is discrete, and let $t \in \Lambda$ such that $\Lambda = \langle t \rangle_{\mathbb{Z}}$. Then $V^- = [\frac{1}{2}(r_1 - t), \frac{1}{2}r_1]$ is a fundamental domain for Γ , modulo boundary. Let $\psi: V' \rightarrow \mathbb{R}$ be any function such that $\psi|_{\partial V'} = 0$. Then the equivariance formula*

$$\psi(\gamma(x)) = \chi(\gamma)\psi(x) \quad \text{for } x \in \mathbb{R} \text{ and } \gamma \in \Gamma \tag{3}$$

gives a well-defined extension of ψ to all of \mathbb{R} . It satisfies equations (2) for all $x, y \in \mathbb{R}$.

Proof. The reflection ρ_{r_1} sends V^- to $V^+ = [\frac{1}{2}r_1, \frac{1}{2}(r_1 + t)]$. Then the translations in T tile all of \mathbb{R} with copies of V^- and V^+ . Thus V^- is a fundamental domain for Γ , modulo the boundary. Since on the boundary, $\psi = 0$, the extension is well-defined. Since $\psi(t_i) = 0$ and $\psi(r_i) = 0$, equation (3) implies (2). \square

Later we will modify the function ψ to become a suitable perturbation $\bar{\pi}$.

3 Proof of the main results

In this section, we assume that the breakpoints B are all rational. Then there exists a $q \in \mathbb{N}$ such that $G = \langle B \rangle_{\mathbb{Z}} = \frac{1}{q}\mathbb{Z}$. Let $\hat{G} = \frac{1}{4q}\mathbb{Z}$. We will think of π as a piecewise linear function with breakpoints in G . We need the following definition.

Definition 3.1. Let π be a minimal valid function.

- (a) For any closed interval $I \subset [0, 1]$, if π is affine in $\text{int}(I)$ and if for all valid functions π^1, π^2 such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$ we have that π^1, π^2 are affine in $\text{int}(I)$, then we say that π is *affine imposing in I* .
- (b) For a collection \mathcal{I} of closed intervals of $[0, 1]$, if for all $I \in \mathcal{I}$, π is affine imposing in I , then we say that π is *affine imposing in \mathcal{I}* .

We either show that π is affine imposing in $\mathcal{I}_{G,1}$ (subsection 3.1) or construct a piecewise linear Γ -equivariant perturbation with breakpoints in \hat{G} that proves π is not extreme (subsection 3.2). If π is affine imposing in $\mathcal{I}_{G,1}$, we set up a system of linear equations to decide if π is extreme or not (subsection 3.3). In subsection 3.4, we prove the main theorems stated in the introduction.

3.1 Imposing affine linearity on open subintervals

For the remainder of this paper, we will use reflections and translations modulo 1 to compensate for the fact that our function is periodic with period 1. Working modulo 1 is accounted for by applying the translation τ_1 whenever needed. Hence, we define the reflection $\bar{\rho}_v(x) = v \ominus x$ and the translation $\bar{\tau}_v(x) = v \oplus x$. Now we define sets that can be thought of as domains for these functions. Let

$$D_{\bar{\rho}_v} = \{x \in [0, 1] \mid \Delta\pi(x, v \ominus x) = \pi(x) + \pi(v \ominus x) - \pi(v) = 0\}$$

and let

$$D_{\bar{\tau}_v} = \{x \in [0, 1] \mid \Delta\pi(x, v) = \pi(x) + \pi(v) - \pi(x \oplus v) = 0\}.$$

Observe that, since G is a group and therefore the breakpoints are evenly spaced, every face of the polyhedral complex \mathcal{P}_G introduced in section 2.3 is a simplex and $\Delta\pi$ is affine in every face of \mathcal{P}_G . See Figure 2. Furthermore, for any $I \in \mathcal{I}_{G,1}$ and $v \in G$, we have that $\bar{\rho}_v(I), \bar{\tau}_v(I) \in \mathcal{I}_{G,1}$.

Let $\mathcal{G} = \mathcal{G}(\mathcal{I}_{G,1}, \mathcal{E})$ be an undirected graph with node set $\mathcal{I}_{G,1}$ and edge set \mathcal{E} where $\{I, J\} \in \mathcal{E}$ if and only if for some $I \in \mathcal{I}_{G,1}$ and $v \in G$ we have either $I \subseteq D_{\bar{\rho}_v}$ and $\bar{\rho}_v(I) = J$, which implies $J \subseteq D_{\bar{\rho}_v}$, or $I \subseteq D_{\bar{\tau}_v}$ and $\bar{\tau}_v(I) = J$, which implies $J \subseteq D_{\bar{\tau}_v}$.

For each $I \in \mathcal{I}_{G,1}$, let \mathcal{G}_I be the connected component of \mathcal{G} containing I . Define the set of 2-dimensional faces of \mathcal{P}_G where $\Delta\pi$ vanishes as

$$\mathcal{P}_G^{0,2} = \{F \in \mathcal{P}_G \mid \Delta\pi(x, y) = 0 \text{ for all } (x, y) \in \text{rel int}(F), \dim(F) = 2\}.$$

Let

$$\mathcal{I}_{G,1}^2 = \{I \in \mathcal{I}_{G,1} \mid I \subseteq (1, 0)F \text{ or } I \subseteq (1, 1)F \text{ for } F \in \mathcal{P}_G^{0,2}\},$$

where $(1, 0)F = \{x \mid (x, y) \in F\}$ and $(1, 1)F = \{x \oplus y \mid (x, y) \in F\}$. We consider the set of these projections of all 2-dimensional faces and any intervals connected to them in the graph \mathcal{G} ,

$$\mathcal{S}_G = \{J \in \mathcal{I}_{G,1} \mid J \in \mathcal{G}_I \text{ for some } I \in \mathcal{I}_{G,1}^2\}.$$

We will use this set to determine if π is affine imposing in $\mathcal{I}_{G,1}$. We first show that the Interval Lemma can be applied to projections of any face $F \in \mathcal{P}_G^{0,2}$.

Lemma 3.2. π is affine imposing in $\mathcal{I}_{G,1}^2$.

Proof. Consider any $I \in \mathcal{I}_{G,1}^2$. By definition, there exists $F \in \mathcal{P}_B^{0,2}$ such that $I \subseteq (1,0)F$ or $I \subseteq (1,1)F$. We will show the result for $I \subseteq (1,0)F$; the proof for $I \subseteq (1,1)F$ is similar. Fix any $x_0 \in \text{int}(I)$. We will show that there exists $c \in \mathbb{R}$ such that $\pi^1(y) = \pi^1(x_0) + c \cdot (y - x_0)$ for all $y \in \text{int}(I)$. This will prove the claim.

Let $y \in \text{int}(I)$. Since $x_0, y \in \text{int}(I)$, there exist points $(p_1^0, p_2^0), (q_1, q_2) \in \text{rel int}(F)$ such that $x_0 = (1,0) \cdot (p_1^0, p_2^0)$ and $y = (1,0) \cdot (q_1, q_2)$. Therefore, we can construct a sequence of closed intervals $U_0, \dots, U_n \subseteq \text{int}(I)$ and another sequence of closed intervals V_0, \dots, V_n such that $U_i \times V_i \subseteq \text{rel int}(F)$ and $(p_1^0, p_2^0) \in U_0 \times V_0$ and $(q_1, q_2) \in U_n \times V_n$ such that $\text{int}(U_{i-1} \times V_{i-1}) \cap \text{int}(U_i \times V_i) \neq \emptyset$, for all $i = 1, \dots, n$. Therefore, we can find a sequence of points $(p_1^1, p_2^1), \dots, (p_1^n, p_2^n)$ such that $(p_1^i, p_2^i) \in \text{int}(U_{i-1} \times V_{i-1}) \cap \text{int}(U_i \times V_i)$. Let $x_i = (1,0) \cdot (p_1^i, p_2^i)$ for all $i = 1, \dots, n$.

Now, since $\Delta\pi(x, y) = 0$ over $\text{rel int}(F)$, we have that $\Delta\pi(u, v) = 0$ for all $(u, v) \in U_i \times V_i$, $i = 0, \dots, n$. This implies $\pi(u) + \pi(v) = \pi(u \oplus v)$ for all $(u, v) \in U_i \times V_i$, and so the same relation holds for π^1 . Using the Interval Lemma, there exists c_i such that $\pi^1(u) = \pi^1(x_i) + c_i \cdot (u - x_i)$, for all $u \in U_i$ and this holds for every $i = 0, \dots, n$. Observe that x_i belongs to the interior of both U_{i-1} and U_i for all $i = 1, \dots, n$. Therefore, we must have $c_{i-1} = c_i$ for all $i = 1, \dots, n$; we take c to be this common value. Now using the relation $\pi^1(y) = \pi^1(x_0) + (\sum_{i=1}^n \pi^1(x_i) - \pi^1(x_{i-1})) + (\pi^1(y) - \pi^1(x_n))$, we find that $\pi^1(y) = \pi^1(x_0) + c \cdot (y - x_0)$. \square

Also intervals connected in the graph have related slopes.

Lemma 3.3. For $\theta = \pi, \pi^1$, or π^2 , if θ is affine in $\text{int}(I)$, and $\{I, J\} \in \mathcal{E}$, then θ is affine in $\text{int}(J)$ as well with the same slope.

Proof. Since θ is affine in $\text{int}(I)$, let $c, b \in \mathbb{R}$ such that $\theta(x) = cx + b$ for $x \in \text{int}(I)$. If $\{I, J\} \in \mathcal{E}$, then one of two cases could happen.

Case 1. $\text{int}(I) \subseteq D_{\bar{\rho}_a}$, $\bar{\rho}_a(I) = J$ for some $a \in G$. Then $\theta(a \ominus x) + \theta(x) = \theta(a)$ for all $x \in \text{int}(J)$. Since $a \ominus x \in \text{int}(I)$ for all $x \in \text{int}(J)$ and $I, J \subseteq (0, 1)$, there exists an \bar{a} such that $\theta(a \ominus x) = c(\bar{a} - x) + b$ for all $x \in \text{int}(J)$. Then $\theta(x) = cx - c\bar{a} + \theta(a)$ for $x \in \text{int}(J)$, and therefore θ is affine in $\text{int}(J)$ with the same slope.

Case 2. $\text{int}(I) \subseteq D_{\bar{\tau}_a}$ and $\bar{\tau}_a(I) = J$ for some $a \in G$. Then $\theta(x \ominus a) + \theta(a) = \theta(x)$ for all $x \in \text{int}(J)$. Again, Since $x \ominus a \in \text{int}(I)$ for all $x \in \text{int}(J)$ and $I, J \subseteq (0, 1)$, there exists an \bar{a} such that $\theta(x \ominus a) = c(x - \bar{a}) + b$ for all $x \in \text{int}(J)$. Then we obtain $\theta(x) = cx - c\bar{a} + \theta(a)$ for $x \in \text{int}(J)$, and therefore θ is affine in $\text{int}(J)$ with the same slope. \square

Theorem 3.4. If $\mathcal{S}_G = \mathcal{I}_{G,1}$, then π is affine imposing in $\mathcal{I}_{G,1}$, and therefore θ is piecewise linear with breakpoints in G for $\theta = \pi^1, \pi^2$. Furthermore, for each interval $I \in \mathcal{I}_{G,1}^2$, θ has the same slope in J for every interval $J \in \mathcal{G}_I$ and $\theta = \pi, \pi^1, \pi^2$.

Proof. From Lemma 3.3, it follows that if π is affine imposing in I and $\{I, J\} \in \mathcal{E}$, then π is affine imposing in J . Let $J \in \mathcal{I}_{G,1} = \mathcal{S}_G$. Then J must be in a connected component containing an interval $I \in \mathcal{I}_{G,1}^2$. By induction then, π is affine imposing in each interval that is a node of the connected component \mathcal{G}_I , and therefore is affine imposing in J . We conclude that π is affine imposing in $\mathcal{I}_{G,1}$. It follows directly from Lemma 3.3 that for each interval $I \in \mathcal{I}_{G,1}^2$, θ has the same slope in J for every interval $J \in \mathcal{G}_I$ and $\theta = \pi, \pi^1, \pi^2$. \square

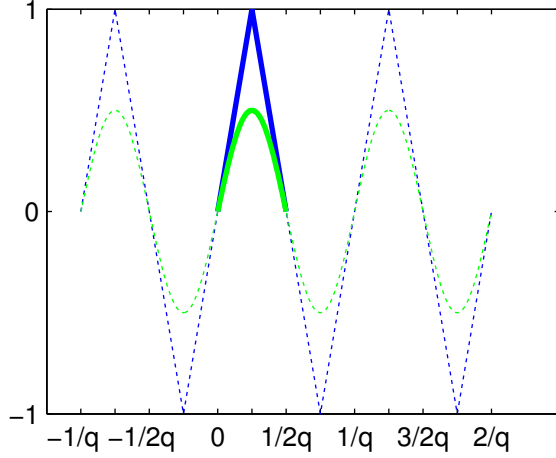


Figure 3: Examples of equivariant perturbations. The continuous piecewise linear function ψ is defined by connecting the points $(0, 0)$, $(\frac{1}{4q}, 1)$, $(\frac{1}{2q}, 0)$ (solid blue lines) and then extended according to Lemma 2.10 to a Γ -equivariant function (dashed blue lines). The light green curve shows the function $\tilde{\psi} = \frac{1}{2} \sin(2\pi qx)$, which is another Γ -equivariant function.

The function in Figure 2 illustrates how intervals can be connected in \mathcal{G} and that intervals with the same slope are not necessarily connected.

3.2 Non-extremality by equivariant perturbation

In this subsection, we will prove the following result.

Lemma 3.5. *Let π be a minimal, piecewise linear function with a set B of rational breakpoints. If $\mathcal{S}_G \neq \mathcal{I}_{G,1}$, then π is not extreme.*

In the proof, we will need an equivariant perturbation that we construct as follows. Let $\Gamma = \langle \rho_g, \tau_g \mid g \in G \rangle$ be the group generated by reflections and translations corresponding to all possible breakpoints, that is, all elements of $G = \frac{1}{q}\mathbb{Z}$. Let the function $\psi: [0, \frac{1}{2q}] \rightarrow \mathbb{R}$ be given by connecting the points $(0, 0)$, $(\frac{1}{4q}, 1)$, $(\frac{1}{2q}, 0)$, and then extending ψ to all of \mathbb{R} using Lemma 2.10. See Figure 3.

Lemma 3.6. *The function $\psi: \mathbb{R} \rightarrow \mathbb{R}$ constructed above has the following properties:*

- (i) $\psi(g) = 0$ for all $g \in G$,
- (ii) $\psi(x) = -\psi(\rho_g(x)) = -\psi(g - x)$ for all $g \in G, x \in [0, 1]$,
- (iii) $\psi(x) = \psi(\tau_g(x)) = \psi(g + x)$ for all $g \in G, x \in [0, 1]$,
- (iv) ψ is piecewise linear with breakpoints in $\hat{G} = \frac{1}{4q}\mathbb{Z}$.

Proof. The properties follow directly from the equivariance formula (3). □

Proof of Lemma 3.5. Let $I \in \mathcal{I}_{G,1} \setminus \mathcal{S}_G$. Let $R = \bigcup_{J \in \mathcal{G}_I} \text{int}(J)$. Let ψ be the Γ -equivariant function of Lemma 3.6. Let

$$\epsilon = \min\{ \Delta\pi_{\hat{F}}(x, y) \neq 0 \mid \hat{F} \in \mathcal{P}_{\hat{G}}, (x, y) \in \text{vert}(\hat{F}) \},$$

and let $\bar{\pi} = \delta_R \cdot \psi$ where δ_R is the indicator function for the set R . We will show that for

$$\pi^1 = \pi + \frac{\epsilon}{3}\bar{\pi}, \quad \pi^2 = \pi - \frac{\epsilon}{3}\bar{\pi},$$

that π^1, π^2 are minimal, and therefore valid functions, and hence π is not extreme. We will show this just for π^1 as the proof for π^2 is the same.

Since $\psi(0) = 0$ and $\psi(f) = 0$, we see that $\pi^1(0) = 0$ and $\pi^1(f) = 1$.

We want to show that π^1 is symmetric and subadditive. We will do this by analyzing the function $\Delta\pi^1(x, y) = \pi^1(x) + \pi^1(y) - \pi^1(x \oplus y)$. Since ψ is piecewise linear over \hat{G} , π^1 is also piecewise linear with breakpoints in \hat{G} , and thus we only need to focus on vertices of faces of $\mathcal{P}_{\hat{G}}$.

Let $I, J, K \in \mathcal{I}_{\hat{G},0} \cup \mathcal{I}_{\hat{G},1}$, such that $\hat{F} = \{(x, y) \mid x \in I, y \in J, x \oplus y \in K\} \in \mathcal{P}_{\hat{G}}$ is non-empty. Let $\Delta\pi_{\hat{F}}^1(u, v) = \pi_I(u) + \pi_J(v) - \pi_K(u \oplus v)$. Let $(u, v) \in \text{vert}(\hat{F})$.

First, if $\Delta\pi_{\hat{F}}(u, v) > 0$, then $\Delta\pi_{\hat{F}}(u, v) \geq \epsilon$ and therefore

$$\Delta\pi_{\hat{F}}^1(u, v) \geq \pi_I(u) - \epsilon/3 + \pi_J(v) - \epsilon/3 - \pi_K(u \oplus v) - \epsilon/3 = \Delta\pi_{\hat{F}}(u, v) - \epsilon \geq 0.$$

Next, we will show that if $\Delta\pi_{\hat{F}}(u, v) = 0$, then $\Delta\pi_{\hat{F}}^1(u, v) = 0$. This will prove two things. First, $\Delta\pi^1(x, y) \geq 0$ for all $x, y \in [0, 1]$, and therefore π^1 is subadditive. Second, since π is symmetric, if $\hat{F} \subset \{(x, y) \mid x \oplus y = f\}$, then $\Delta\pi_{\hat{F}}(x, y) = 0$ for all $(x, y) \in \text{vert}(\hat{F})$, which would imply that $\Delta\pi_{\hat{F}}^1(x, y) = 0$ for all $(x, y) \in \text{vert}(\hat{F})$, proving π^1 is symmetric.

Suppose that $\Delta\pi_{\hat{F}}(u, v) = 0$. We will proceed by cases. See Figure 4 for an illustration of these cases.

Case 1. Suppose $u, v, u \oplus v \notin R$. Then $\delta_R(u) = \delta_R(v) = \delta_R(u \oplus v) = 0$, and $\Delta\pi_{\hat{F}}^1(u, v) = \Delta\pi_{\hat{F}}(u, v) \geq 0$.

Case 2. Suppose $u, v \in \frac{1}{2q}\mathbb{Z}$. Then $u \oplus v \in \frac{1}{2q}\mathbb{Z}$ and, by Lemma 3.6 (i), $\psi(u) = \psi(v) = \psi(u \oplus v) = 0$. Thus $\Delta\pi_{\hat{F}}^1(u, v) = \Delta\pi_{\hat{F}}(u, v) \geq 0$.

Case 3. Suppose we are not in cases 1 or 2. That is, suppose $\Delta\pi_{\hat{F}}(u, v) = 0$, not both u, v are in $\frac{1}{2q}\mathbb{Z}$, and at least one of $u, v, u \oplus v$ is in R . Since $\Delta\pi^1(x, y)$ is symmetric in x and y , without loss of generality, since not both u, v are in $\frac{1}{2q}\mathbb{Z}$, we will assume that $u \notin \frac{1}{2q}\mathbb{Z}$. Therefore, $u \in R \setminus \frac{1}{2q}\mathbb{Z}$.

Since $u \notin \frac{1}{2q}\mathbb{Z}$, $(u, v) \notin \text{vert}(\mathcal{P}_G)$. Therefore, there exists a 1-dimensional or 2-dimensional face $F \in \mathcal{P}_G$ such that $(u, v) \in \text{rel int}(F)$. Since $\hat{F} \subseteq F$, and $\Delta\pi$ is affine in $\text{rel int}(F)$, we see $\Delta\pi_F(x, y) = \Delta\pi_{\hat{F}}(x, y)$. Since π is subadditive, $\Delta\pi_F(x, y) \geq 0$ for all $(x, y) \in \text{rel int}(F)$, and $\Delta\pi_{\hat{F}}(u, v) = 0$ with $(u, v) \in \text{rel int}(\hat{F})$, and therefore $\Delta\pi_{\hat{F}} \equiv 0$. If F were a 2-dimensional face, then $u \in I$ for some $I \in \mathcal{I}_{G,1}^2$, which is a contradiction since $u \in R$. Therefore, $F \in \mathcal{P}_G$ is a 1-dimensional face and hence a subset of one of the three following hyperplanes: $x = u$, $y = v$, or $x + y = u \oplus v$.

Since $u \notin \frac{1}{2q}\mathbb{Z} \supset G$, F cannot be a subset of $x = u$ because it is not a defining hyperplane of the polyhedral complex \mathcal{P}_G . Observe that for any $x \in [0, 1]$ with $x \notin G$, there is a unique

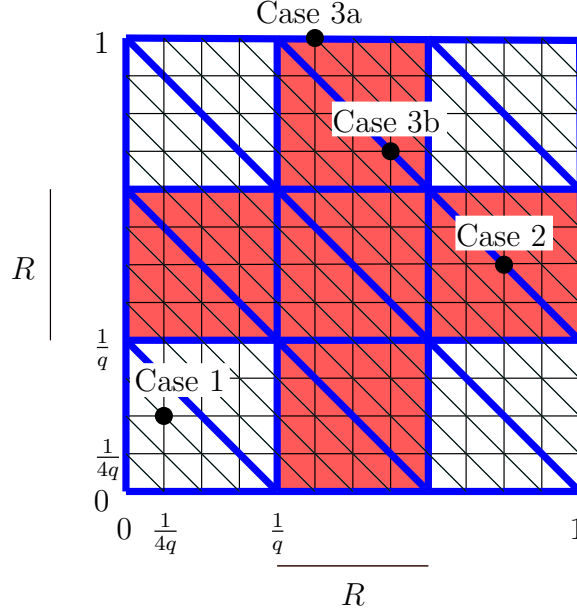


Figure 4: The cases within the proof of Lemma 3.5. The union of open sets R is shaded red. The thicker blue lines plot \mathcal{P}_G , while all of the lines plot $\mathcal{P}_{\hat{G}}$. In Case 1, $u, v, u \oplus v \notin R$. In Case 2, $u, v \in \frac{1}{2q}\mathbb{Z}$. In Case 3, we are not in Cases 1 or 2. Furthermore, in Case 3a, $F \subset \{(x, y) \mid y = v\}$ and $v \in G$, and in Case 3b, $F \subset \{(x, y) \mid x \oplus y = u \oplus v\}$ and $u \oplus v \in G$.

$I_x \in \mathcal{I}_{G,1}$ such that $x \in I_x$. Since $u \notin G$, there exists a unique interval $I_u \in \mathcal{I}_{G,1}$ such that $u \in I$. There are two possible cases.

Case 3a. $F \subset \{(x, y) \mid y = v\}$ and $v \in G$.

Since $v \in G, u \notin \frac{1}{2q}\mathbb{Z}$, it follows that $u \oplus v \notin G$, and there is a unique interval $I_{u \oplus v} \in \mathcal{I}_{G,1}$ containing $u \oplus v$. Since $\Delta\pi_F \equiv 0$, $I_u \subseteq D_{\bar{\tau}_v}$ and $\bar{\tau}_v(I_u) = I_{u \oplus v}$. Therefore $\{I_u, I_{u \oplus v}\} \in \mathcal{E}$ and $\delta_R(u) = \delta_R(u \oplus v)$. Since $v \in G$, we have $\psi(v) = 0$ and $\psi(u) = \psi(\bar{\tau}_v(u)) = \psi(u \oplus v)$ by Lemma 3.6 (iii). It follows that $\bar{\pi}(u) + \bar{\pi}(v) - \bar{\pi}(u \oplus v) = 0$, and therefore $\Delta\pi_{\hat{F}}^1(u, v) = \Delta\pi_{\hat{F}}(u, v) = 0$.

Case 3b. $F \subset \{(x, y) \mid x \oplus y = u \oplus v\}$ and $u \oplus v \in G$.

Since $u \oplus v \in G, u \notin \frac{1}{2q}\mathbb{Z}$, it follows that $v \notin G$, and there is a unique interval $I_v \in \mathcal{I}_{G,1}$ containing v . Since $\Delta\pi \equiv 0$ in $\text{rel int}(F)$, $I_u \subseteq D_{\bar{\rho}_v}$ and $\bar{\rho}_v(I_u) = I_v$. Therefore $\{I_u, I_v\} \in \mathcal{E}$ and $\delta_R(u) = \delta_R(v)$. Since $u \oplus v \in G$, $\psi(u) = -\psi(\bar{\rho}_{u \oplus v}(u)) = -\psi(v)$ by Lemma 3.6 (ii). It follows that $\bar{\pi}(u) + \bar{\pi}(v) - \bar{\pi}(u \oplus v) = 0$, and therefore $\Delta\pi_{\hat{F}}^1(u, v) = \Delta\pi_{\hat{F}}(u, v) = 0$.

We conclude that π^1 (and similarly π^2) is subadditive and symmetric, and therefore minimal and hence valid. Therefore π is not extreme. \square

Remark 3.7. To show that π is not extreme in the above lemma, the perturbation function ψ need not be piecewise linear. This choice was made to simplify the proof. In fact, any Γ -equivariant function $\tilde{\psi} \neq 0$ constructed with Lemma 2.10 with $|\tilde{\psi}| < |\psi|$ suffices. This is depicted in Figure 3.

However, the specific form of our function ψ as a piecewise linear function with breakpoints in \hat{G} (Lemma 3.6 (iv)) implies the following corollary.

Corollary 3.8. *If π is not affine imposing over $\mathcal{I}_{G,1}$, then there exist distinct minimal π^1, π^2 that are piecewise linear with breakpoints in \hat{G} such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$.*

3.3 Extremality and non-extremality by a system of linear equations

In this section we suppose π is a minimal piecewise linear function with breakpoints in B that is affine imposing in $\mathcal{I}_{B,1}$. Therefore, π^1 and π^2 must also be piecewise linear functions with breakpoints in B . It is clear that whenever $\pi(x) + \pi(y) = \pi(x \oplus y)$, the functions π^1 and π^2 must also satisfy this equality relation, that is, $\pi^i(x) + \pi^i(y) = \pi^i(x \oplus y)$. Let $\Delta\pi^i(x, y) = \pi^i(x) + \pi^i(y) - \pi^i(x \oplus y)$ for $i = 1, 2$. Equivalently, whenever $\Delta\pi(x, y) = 0$, we also have $\Delta\pi^i(x, y) = 0$ for $i = 1, 2$. This is easy to see since π, π_1, π_2 are subadditive, $\Delta\pi, \Delta\pi^1, \Delta\pi^2 \geq 0$ and $\Delta\pi = \frac{1}{2}\Delta\pi^1 + \frac{1}{2}\Delta\pi^2$. We extend this idea slightly.

Lemma 3.9. *Let $F \in \mathcal{P}_B$ and let $(u, v) \in \text{vert}(F)$. If $\Delta\pi_F(u, v) = 0$ then $\Delta\pi_F^i(u, v) = 0$ for $i = 1, 2$.*

Proof. From definition of $\Delta\pi^1, \Delta\pi^2$, we see that $\Delta\pi = \frac{1}{2}\Delta\pi^1 + \frac{1}{2}\Delta\pi^2$. Since π^1, π^2 are minimal, $\Delta\pi^i \geq 0$ for $i = 1, 2$. If $\Delta\pi_F(u, v) = 0$, then

$$0 = \Delta\pi_F(u, v) = \lim_{\substack{(x,y) \rightarrow (u,v) \\ (x,y) \in \text{rel int}(F)}} \Delta\pi(x, y) = \lim_{\substack{(x,y) \rightarrow (u,v) \\ (x,y) \in \text{rel int}(F)}} \frac{1}{2}\Delta\pi^1(x, y) + \frac{1}{2}\Delta\pi^2(x, y).$$

Since the right hand side limit is zero and $\Delta\pi^1, \Delta\pi^2 \geq 0$, we must have that

$$0 = \lim_{\substack{(x,y) \rightarrow (u,v) \\ (x,y) \in \text{rel int}(F)}} \Delta\pi^1(x, y) = \lim_{\substack{(x,y) \rightarrow (u,v) \\ (x,y) \in \text{rel int}(F)}} \Delta\pi^2(x, y),$$

and therefore $\Delta\pi_F^i(u, v) = 0$ for $i = 1, 2$. □

We now set up a system of linear equations that π satisfies and that π_1 and π_2 must also satisfy. Let φ be an arbitrary piecewise linear function with breakpoints in B such that $\varphi(x) = m_I x + b_I$ for $x \in I \in \mathcal{I}_{B,1}$ and $\varphi(x) = b_I$ for all $x \in I \in \mathcal{I}_{B,0}$. For every $F \in \mathcal{P}_B$, let $\Delta\varphi_F(x, y) = \varphi_I(x) + \varphi_J(y) - \varphi_K(x \oplus y)$, where $F = \{(x, y) \mid x \in I, y \in J, x \oplus y \in K\}$ and $I, J, K \in \mathcal{I}_{B,1} \cup \mathcal{I}_{B,0}$. Suppose φ satisfies the following system of linear equations in terms of m_I, b_I for $I \in \mathcal{I}_{B,1}$ and b_I for $I \in \mathcal{I}_{B,0}$:

$$\begin{cases} \varphi(0) = 0, \\ \varphi(f) = 1, \\ \varphi(1) = 0, \\ \Delta\varphi_F(u, v) = 0 \text{ whenever } \Delta\pi_F(u, v) = 0 \text{ for all } (u, v) \in \text{vert}(F) \text{ and } F \in \mathcal{P}_B. \end{cases} \quad (4)$$

Since π exists and satisfies the system of equations, we know that the system has a solution. We will show that this solution is unique if and only if π is extreme.

Theorem 3.10. *If π is a piecewise linear valid function with breakpoints in B and the system of equations (4) does not have a unique solution, then π is not extreme.*

Proof. On the other hand, suppose (4) does not have a unique solution. Let $\{(\bar{m}_I, \bar{b}_I)_{I \in \mathcal{I}_{B,1}}, (\bar{b}_I)_{I \in \mathcal{I}_{B,0}}\}$ be a non-trivial element in the kernel of the system above. Let $\bar{\varphi}$ be the piecewise linear function given by $\bar{\varphi}(x) = \bar{m}_I x + \bar{b}_I$ for $x \in I \in \mathcal{I}_{B,1}$ and $\bar{\varphi}(x) = \bar{b}_I$ for all $x \in I \in \mathcal{I}_{B,0}$. Then for any ϵ , $\pi + \epsilon \bar{\varphi}$ also satisfies the system of equations. Let

$$\epsilon = \min\{ \Delta\pi_F(x, y) \neq 0 \mid F \in \mathcal{P}_B, (x, y) \in \text{vert}(F) \}.$$

and set $\pi^1 = \pi + \frac{\epsilon}{3\|\bar{\varphi}\|_\infty} \bar{\varphi}$, $\pi^2 = \pi - \frac{\epsilon}{3\|\bar{\varphi}\|_\infty} \bar{\varphi}$. Note that $0 < \|\bar{\varphi}\|_\infty < \infty$ since $\bar{\varphi}$ comes from a non-trivial element in the kernel, and because it is piecewise linear on a compact domain. We claim that π^1, π^2 are both minimal. As before, we show this for π^1 , and π^2 is similar. Since π satisfies the system (4) and $\bar{\varphi}$ is an element of the kernel, π^1 satisfies the system (4) as well. In particular, we have $\pi^1(0) = 0, \pi^1(f) = 1, \pi^1(1) = 0$.

Next, π^1 is symmetric because the symmetry conditions are implied here, that is, since we require that $\varphi(f) = 1$, and since π is minimal, $\Delta\pi_F \equiv 0$ whenever $F \subseteq \{(x, y) \mid x \oplus y = f\}$, and therefore $\Delta\varphi_F(u, v) = 0$ is an equation in (4) for each $(u, v) \in \text{vert}(F)$.

Lastly, we show that π^1 is subadditive. Let $F \in \mathcal{P}_B$ and $(u, v) \in \text{vert}(F)$. If $\Delta\pi_F(u, v) = 0$, then $\Delta\varphi_F(u, v) = 0$, as implied by the system of equations. Otherwise, if $\Delta\pi_F(u, v) > 0$, then

$$\begin{aligned} \Delta\pi_F^1(u, v) &= \Delta\pi_F(u, v) + \frac{\epsilon}{3\|\bar{\varphi}\|_\infty} \bar{\varphi}(u) + \frac{\epsilon}{3\|\bar{\varphi}\|_\infty} \bar{\varphi}(v) - \frac{\epsilon}{3\|\bar{\varphi}\|_\infty} \bar{\varphi}(u \oplus v) \\ &\geq \Delta\pi_F(u, v) - \frac{\epsilon}{3} - \frac{\epsilon}{3} - \frac{\epsilon}{3} \geq 0 \end{aligned}$$

Therefore, by Theorem 2.5, π^1 (and π^2) is subadditive and therefore minimal and valid. Therefore π is not extreme. \square

Corollary 3.11. *If π is a minimal piecewise linear function with breakpoints in B and is affine imposing in $\mathcal{I}_{B,1}$, then π is extreme if and only if the system of equations (4) has a unique solution.*

Proof. Suppose there exist distinct, valid functions π^1, π^2 such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$. Since π is minimal and affine imposing in $\mathcal{I}_{B,1}$, π^1, π^2 are minimal piecewise linear functions with the same breakpoints B . Furthermore, π and, by Lemma 3.9, also π^1, π^2 satisfy the system of equations (4). If this system has a unique solution, then $\pi = \pi^1 = \pi^2$, which is a contradiction since π^1, π^2 were assumed distinct. Therefore π is extreme.

On the other hand, if the system (4) does not have a unique solution, then by Theorem 3.10, π is not extreme. \square

Theorem 3.12. *Let π be a piecewise linear minimal valid function (possibly discontinuous) whose breakpoints are rational with least common denominator q . Then π is extreme if and only if the system of equations (4) with $B = \hat{G} = \frac{1}{4q}\mathbb{Z}$ has a unique solution.*

Proof. The forward direction is the contrapositive of Theorem 3.10, applied to the set of breakpoints $B = \hat{G}$. For the reverse direction, let $G = \frac{1}{q}\mathbb{Z}$. By the contrapositive of Corollary 3.8, π is affine imposing in $\mathcal{I}_{G,1}$. Then π is also affine imposing on $\mathcal{I}_{\hat{G},1}$ since it is a finer interval set. By Corollary 3.11, since π is affine imposing in $\mathcal{I}_{\hat{G},1}$ and the system of equations (4) with $B = \hat{G}$ has a unique solution, π is extreme. \square

Remark 3.13 (Reduced System). Writing down a reduced system is advantageous for reading a proof of a function being extreme. There are two main ways to do this, as have been done in previous literature.

First, if π is continuous, the variables b_I for any $I \in \mathcal{I}_{B,0}$ become redundant and can be removed from the system. Also, it follows that π_1, π_2 also must be continuous, so we can remove the variables b_I for all $I \in \mathcal{I}_{B,1}$ and replace these values as integrals over the function, which are linear in the slopes c_I for $I \in \mathcal{I}_{B,1}$.

Second, the Interval Lemma guarantees that certain intervals, such as intervals connected in the graph $\mathcal{G}(\mathcal{I}_{G,1}, \mathcal{E})$, actually have the same slope. Therefore, the number of variables can be reduced to the number of connected components of intervals. This technique was first applied to prove the two slope theorem [19].

3.4 Proofs of Theorems 1.3 and 1.5

We are now ready to present the proofs of Theorems 1.3 and 1.5.

Proof of Theorem 1.3. Given the piecewise linear function π , the algorithm performs the following test. Test if the system (4) with $B = \hat{G} = \frac{1}{4q}\mathbb{Z}$ has a unique solution, where q is the least common denominator of the breakpoints of π . If yes, then report that π is extreme; else, report that π is not extreme. Theorem 3.12 guarantees the correctness of this algorithm. \square

Proof of Theorem 1.5. We first show that if π is extreme, then $\pi|_{\hat{G}}$ is extreme for $R_f(\hat{G}, \mathbb{Z})$. If not, then there exist two distinct minimal functions $\bar{\pi}^1, \bar{\pi}^2$, both functions from \hat{G} to \mathbb{R}_+ , such that $\pi|_{\hat{G}} = \frac{1}{2}\bar{\pi}^1 + \frac{1}{2}\bar{\pi}^2$. Let π^i be the linear interpolation of $\bar{\pi}^i$, $i = 1, 2$. It can be verified that π^i is minimal because $\bar{\pi}^i$ is minimal. This contradicts the extremality of π .

Now we show that if $\pi|_{\hat{G}}$ is extreme for $R_f(\hat{G}, \mathbb{Z})$, then π is extreme. If π is not extreme then by Theorem 3.12 the system of equations (4) does not have a unique solution. Then we can construct π^1 and π^2 as in the proof of Theorem 3.10 using the non-trivial element in the kernel of (4), such that $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$. Since π is a continuous piecewise linear function, $\limsup_{h \rightarrow 0} |\frac{\pi(h)}{h}| < \infty$. Theorem 2.7 then tells us that π^1 and π^2 both have to be continuous, and by construction have breakpoints in \hat{G} . Thus, since $\pi^1 \neq \pi^2$, there exists a breakpoint $v \in \hat{G}$ such that $\pi^1(v) \neq \pi^2(v)$. Thus, $\pi^1|_{\hat{G}}$ and $\pi^2|_{\hat{G}}$ are two distinct minimal valid functions for $R_f(\hat{G}, \mathbb{Z})$. Moreover, since $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, we have that $\pi|_{\hat{G}} = \frac{1}{2}\pi^1|_{\hat{G}} + \frac{1}{2}\pi^2|_{\hat{G}}$. This contradicts the extremality of $\pi|_{\hat{G}}$. \square

4 The irrational case: A new principle for proving extremality

This section is motivated by the function in Figure 5. This function has three slopes, one of which is zero, and has translation points a_0, a_1, a_2 such that $\pi(a_i) + \pi(x) = \pi(a_i \oplus x)$ for $i = 0, 1, 2$ and x in a certain interval $[A, A_i]$. When certain parameters are chosen appropriately, we will show that this function is extreme. In doing so, we showcase a new type of proof for a function to be extreme. In this section, we will primarily do arithmetic in \mathbb{R} , and we will be careful to stay inside the interval $[0, 1]$.

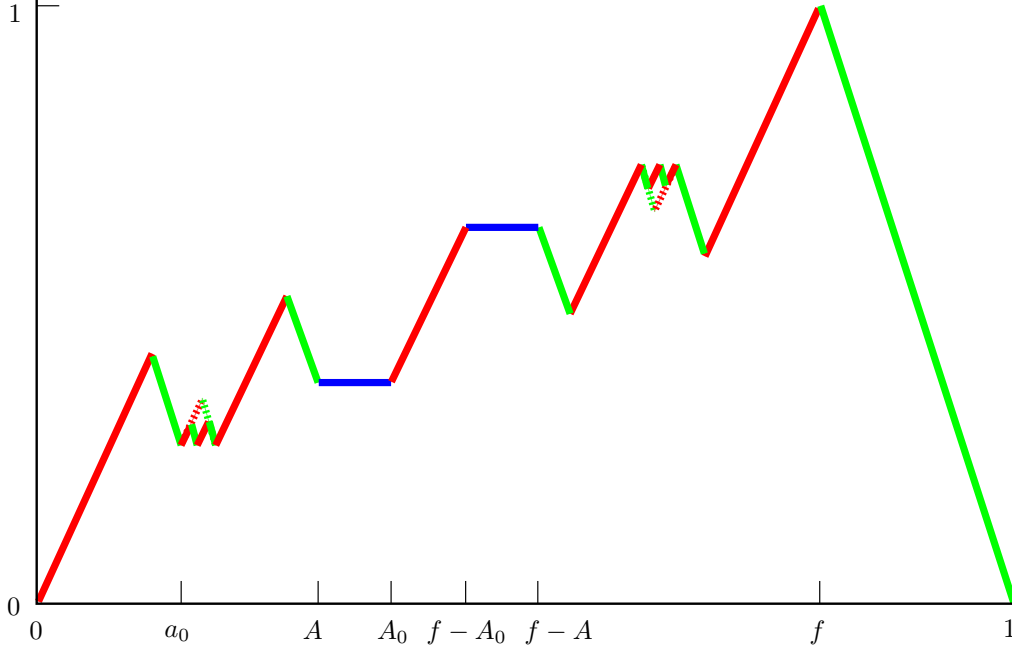


Figure 5: This function π is extreme. To show this, we use three translations that combine to create a dense set of relations that cause π to be affine imposing where π has slope 0. This argument crucially relies on the fact that one of the translations is irrational. Using standard techniques, we show π is affine imposing on all other intervals and then show that π is the unique solutions to a system of equations. Surprisingly, if all the points are rational, then this function is not extreme. We construct this function in two steps. In Step 1, we describe the function above with dotted lines. In Step 2, we add on the two extra zig-zags. See Figure 6 for more details of this function.

4.1 Function requirements

Here we explain restrictions that we require of some of the breakpoints of our function; see also Figure 6.

Assumption 4.1. *Let $a_0, a_1, a_2, t_1, t_2, f, A, A_0, A_1, A_2 \in (0, 1)$ such that the following hold:*

- (i) *The numbers t_1, t_2 are linearly independent over \mathbb{Q} .*
- (ii) *We have $a_1 = a_0 + t_1, a_2 = a_0 + t_2$, and $0 < a_0 < a_1 < a_2 < A < f/2$,*
- (iii) *We have $a_i + A = f - A_i$ and $A_0 > A_1 > A_2 \geq \frac{A_0 + A}{2} > A \geq 0$.*

Let x_0 be the midpoint of $[A, A_0]$, that is, $x_0 = (A + A_0)/2$. Observe that, for $i = 1, 2$,

$$t_i = a_i - a_0 = A_0 - A_i < A_0 - \frac{A_0 + A}{2} = \frac{A_0 - A}{2}.$$

We then have $x_0 \pm t_i \in [A, A_0]$ for $i = 1, 2$.

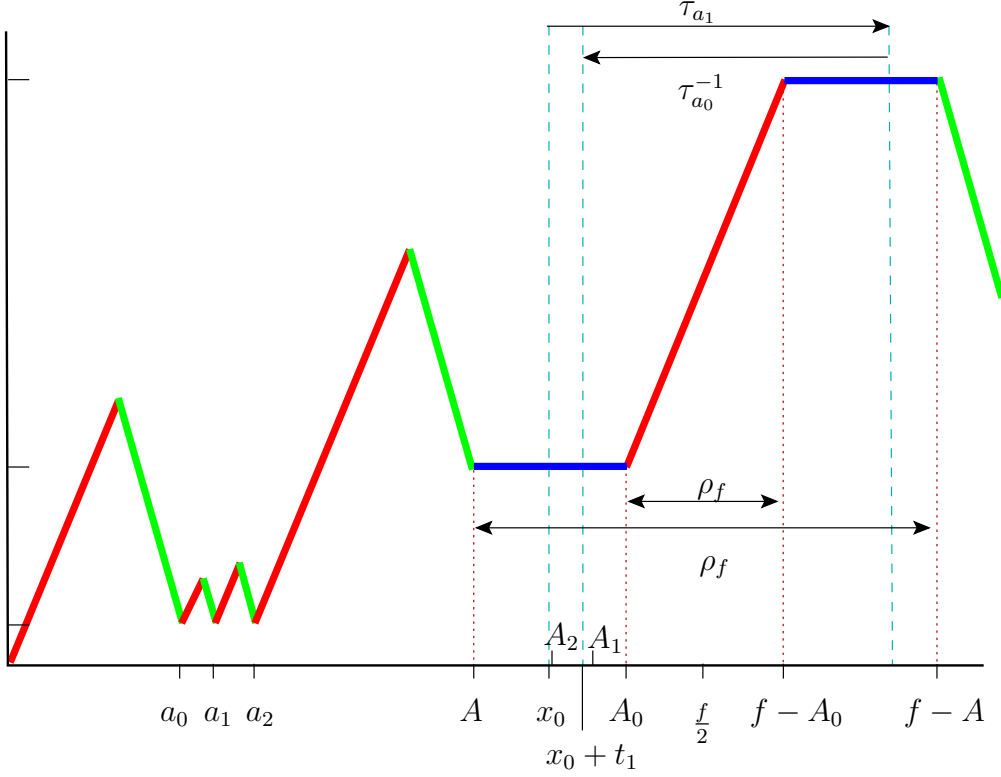


Figure 6: A zoomed in picture of Figure 5 to show in further detail the variables discussed in this section. For our construction to work, we choose a_i and $f - a_i$, for either $i = 1$ or $i = 2$, to be irrational numbers, and all other breakpoints to be rational. It follows that $t_1 = a_1 - a_0$ and $t_2 = a_2 - a_0$ are linearly independent over \mathbb{Q} . The arrows indicate the action of the translations τ_{a_1} , $\tau_{a_0}^{-1}$ and the reflection ρ_f .

Let $\Gamma = \langle \rho_f, \tau_{a_0}, \tau_{a_1}, \tau_{a_2} \rangle$, as defined in section 2.5, and consider the orbit

$$X = \Gamma(x_0) = \{ \gamma(x_0) \mid \gamma \in \Gamma \}.$$

From Lemma 2.9, $\Gamma = T \rtimes \langle \rho_f \rangle$ where $T = \langle \tau_{a_0}, \tau_{a_1}, \tau_{a_2} \rangle$. Let $\tau_a \in T$ be a translation and observe that

$$\tau_{a_0}(\gamma(x_0)) = a_0 + (x + a) = (a_0 + x_0) + a = (f - x_0) + a = f - (x_0 - a) = \rho_f(\gamma^{-1}(x_0)).$$

Therefore, the translation τ_{a_0} is redundant in this orbit and we can rewrite X with one fewer translation as $X = \{ \gamma(x_0) \mid \gamma \in \langle \rho_f, \tau_{a_0}^{-1} \tau_{a_1}, \tau_{a_0}^{-1} \tau_{a_2} \rangle \}$, or more simply as

$$X = (x_0 + \Lambda) \cup (\rho_f(x_0) + \Lambda)$$

where $\Lambda = \langle t_1, t_2 \rangle_{\mathbb{Z}}$. The key element here is that Λ is dense in \mathbb{R} because t_1, t_2 are linearly independent over \mathbb{Q} .

For the same reason, there is bijection from $x_0 + \Lambda$ to \mathbb{Z}^2 as $x_0 + \lambda_1 t_1 + \lambda_2 t_2 \mapsto (\lambda_1, \lambda_2)$. Let $\ell: x_0 + \Lambda \rightarrow \mathbb{N}$ by $x \mapsto |\lambda_1| + |\lambda_2|$. This map is well defined because of the bijection between $x_0 + \Lambda$ and \mathbb{Z}^2 .

Recall now that if π is a minimal valid function and $\pi = \frac{1}{2}\pi^1 + \frac{1}{2}\pi^2$, then π, π^1, π^2 are minimal valid functions as well. Furthermore, from Lemma 3.9, if $\pi(x) + \pi(y) = \pi(x + y)$, then $\pi^i(x) + \pi^i(y) = \pi^i(x + y)$ for $i = 1, 2$. Thus the difference $\bar{\pi} = \pi^1 - \pi$ satisfies also $\bar{\pi}(x) + \bar{\pi}(y) = \bar{\pi}(x + y)$.

Lemma 4.2. *Suppose that for all $x \in [A, A_i]$, $\bar{\pi}(a_i) + \bar{\pi}(x) = \bar{\pi}(a_i + x)$ for $i = 0, 1, 2$. If $x = x_0 + \lambda_1 t_1 + \lambda_2 t_2 \in (x_0 + \Lambda) \cap [A, A_0]$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$, then*

$$\bar{\pi}(x) - \bar{\pi}(x_0) = \lambda_1 (\bar{\pi}(a_1) - \bar{\pi}(a_0)) + \lambda_2 (\bar{\pi}(a_2) - \bar{\pi}(a_0)).$$

Proof. Our proof will be by induction on $\ell(x)$. For $\ell(x) = 0$, we have $\lambda_1 = \lambda_2 = 0$, thus $x = x_0$ and the result is trivial.

Now suppose $\ell(x) > 0$. We will show the result for $x \in (x_0, A_0]$ as the proof for $x \in [A, x_0]$ is similar. Since $x > x_0$, we must have $\lambda_i > 0$ for either $i = 1$ or $i = 2$. Without loss of generality, let $i = 1$. Note that $x \in (x_0, A_0] \subset [A, A_0]$, and therefore

$$\bar{\pi}(a_0) + \bar{\pi}(x) = \bar{\pi}(x + a_0).$$

Consider the point $x - t_1 = x + a_0 - a_1 = x - A_0 + A_1$. First, since $x \leq A_0$, we have $x - t_1 \leq A_0 - A_0 + A_1 = A_1$. Second, since $x > x_0$, we have $x - t_1 > x_0 - t_1 \geq A$. Therefore $x - t_1 \in [A, A_1]$ and

$$\bar{\pi}(a_1) + \bar{\pi}(x - t_1) = \bar{\pi}(x + a_0).$$

Subtracting these two equations and rearranging terms yields

$$\bar{\pi}(x) - \bar{\pi}(x - t_1) = \bar{\pi}(a_1) - \bar{\pi}(a_0).$$

Since $x - t_1 \in [A, A_0]$ and $\ell(x - t_1) = |\lambda_1 - 1| + |\lambda_2| = |\lambda_1| - 1 + |\lambda_2| = \ell(x) - 1$, the induction hypothesis holds for $x - t_1$ and hence

$$\bar{\pi}(x - t_1) - \bar{\pi}(x_0) = (\lambda_1 - 1)(\bar{\pi}(a_1) - \bar{\pi}(a_0)) + \lambda_2(\bar{\pi}(a_2) - \bar{\pi}(a_0)).$$

Therefore

$$\begin{aligned} \bar{\pi}(x) - \bar{\pi}(x_0) &= (\bar{\pi}(x) - \bar{\pi}(x - t_1)) + (\bar{\pi}(x - t_1) - \bar{\pi}(x_0)) \\ &= (\bar{\pi}(a_1) - \bar{\pi}(a_0)) + ((\lambda_1 - 1)(\bar{\pi}(a_1) - \bar{\pi}(a_0)) + \lambda_2(\bar{\pi}(a_2) - \bar{\pi}(a_0))) \\ &= \lambda_1 (\bar{\pi}(a_1) - \bar{\pi}(a_0)) + \lambda_2 (\bar{\pi}(a_2) - \bar{\pi}(a_0)). \end{aligned}$$

As stated before, the proof for $x \in [A, x_0]$ is similar and is done by supposing $\lambda_1 < 0$ and considering the point $x + a_1 - a_0 = x - t_1$. The calculations are very similar. \square

4.2 Construction

We now give a precise definition of the function in Figure 5, and then apply the above lemma. This function will have three slopes, c_1, c_2, c_3 , where we choose $c_2 = 0$. The construction will be done in two steps. We leave this construction general because, although we will only show it is extreme for one choice of parameters, it is indeed extreme for many choices.

Step 1. We will determine variables d_j^i , signifying the i^{th} interval of slope c_j . The intervals written in order have the following lengths:

$$d_1^1, d_3^1, d_1^2, d_3^2, d_2^1, d_1^3, d_2^2, d_3^3, d_1^4, d_3^4, d_1^5, 1 - f,$$

where $a_0 = d_1^1 + d_3^1$, $A = a_0 + d_1^2 + d_3^2$, $A_0 = A + d_2^1$, and $f - A = A + d_2^2 + d_1^3$. Note that the interval $[A, A_0]$ has slope $c_2 = 0$ in this notation.

We begin by picking $f \in (0, 1)$. On the interval $[f, 1]$, the function will have only slope c_3 . We divide up the length of the interval $[0, f]$ into lengths d_1, d_2, d_3 that will be the amount of $[0, f]$ with slopes c_1, c_2, c_3 , respectively. So $d_1 + d_2 + d_3 = f$. Therefore we have the following equations:

$$c_1 = \frac{1 - d_2 c_2 - d_3 c_3}{d_1}, \quad c_2 = 0, \quad c_3 = \frac{-1}{1 - f}, \quad d_1 + d_2 + d_3 = f.$$

Now we subdivide each of these lengths into smaller lengths. Inside $[0, f]$, we want 5 intervals with slope c_1 , 2 intervals with slope c_2 and 4 intervals with slope c_3 . Therefore we give ourselves the following positive variables

$$\begin{aligned} d_1^1 + d_1^2 + d_1^3 + d_1^4 + d_1^5 &= d_1, \\ d_2^1 + d_2^2 &= d_2, \\ d_3^1 + d_3^2 + d_3^3 + d_3^4 &= d_3. \end{aligned}$$

To preserve symmetry in the function, we restrict ourselves to

$$d_1^1 = d_1^5, \quad d_1^2 = d_1^4, \quad d_2^1 = d_2^2, \quad d_3^1 = d_3^4, \quad d_3^2 = d_3^3.$$

We choose our parameters such that

$$\pi(a_0) + \pi(A) = \pi(f - A) \quad \text{and} \quad a_0 + A = f - A,$$

which gives the two equations

$$d_1^1 + d_3^1 = d_2^1 + d_1^3, \quad d_1^1 c_1 + d_3^1 c_3 = d_2^1 c_2 + d_1^3 c_3.$$

By choosing values for the variables f, d_1, d_2 , and $d_1^1 + d_3^1$, the remaining variables can be determined uniquely.

Step 2. We will create 2 more additivity equations:

$$\pi(a_1) + \pi(A) = \pi(a_1 + A) \quad \text{and} \quad \pi(a_2) + \pi(A) = \pi(A + a_2).$$

For this, we pick positive δ^1, δ^2 such that the sum of these values is less than $d_2^1/2$. We set $t_1 = \delta^1$, $t_2 = \delta^1 + \delta^2$, $a_1 = a_0 + t_1$ and $a_2 = a_0 + t_2$. For each δ^i , find δ_1^i, δ_3^i such that

$\delta_1^i + \delta_3^i = \delta^i$ and $c_1\delta_1^i + c_3\delta_3^i = 0$. With this, we force $\pi(a_0) = \pi(a_1) = \pi(a_2)$, and the desired additivity relations follow.

Now we adjust some previous parameters to make room for new zig-zags:

$$\tilde{d}_1^2 = d_1^2 - \delta_1^1 - \delta_1^2, \quad \tilde{d}_1^4 = d_1^4 - \delta_1^1 - \delta_1^2, \quad \tilde{d}_3^2 = d_3^2 - \delta_3^1 - \delta_3^2, \quad \tilde{d}_3^3 = d_3^3 - \delta_3^1 - \delta_3^2.$$

Now the intervals written in order have the following lengths:

$$d_1^1, d_3^1, \delta_1^1, \delta_3^1, \delta_1^2, \delta_3^2, \tilde{d}_1^2, \tilde{d}_3^2, d_2^1, d_1^3, d_2^2, \tilde{d}_3^3, \tilde{d}_1^4, \delta_3^2, \delta_1^2, \dots, \delta_3^1, \delta_1^1, d_3^4, d_1^5, e_3,$$

where each interval has slope corresponding to the subscript. Let $t_1 = \delta^1, t_2 = \delta^1 + \delta^2$, and let $a_1 = a_0 + t_1, a_2 = a_0 + t_2$. The parameters δ^1, δ^2 will be chosen such that t_1, t_2 are linearly independent over \mathbb{Q} .

4.3 Proof of Extremality

Lemma 4.3. *Let $f = 4/5, d_1 = 3/5, d_3 = 1/10, d_1^1 + d_3^1 = 15/100, \delta^1 = 1/200, \delta^2 = \sqrt{2}/200$ and let π be the function given by these parameters and the construction above. Then π is extreme.*

Proof. Using Theorem 2.5, π can be shown that π is a minimal function. Furthermore, these parameters satisfy Assumption 4.1 and π satisfies the hypotheses of Lemma 4.2. Let π_1, π_2 be minimal valid functions such that $\pi = \pi_1 + \pi_2$. From Theorem 2.7, it follows that $\bar{\pi} = \pi^1 - \pi$ is continuous, and since π, π_1, π_2 are minimal, $\bar{\pi}(0) = 0$.

Let $I \in \mathcal{I}_{B,1}$ be an interval with slope c_1 and let $x \in I$. Since I is open, there exists an $\epsilon > 0$ such that $U = [x - \epsilon/2, x + \epsilon/2] \subset I$ and $V = [0, \epsilon] \subset [0, d_1^1]$. Since $\pi(u) + \pi(v) = \pi(u+v)$ for all $u \in U, v \in V$, by the Interval Lemma, π is affine imposing in $U, V, U+V$ and the slopes are connected. Repeating this for every point x that has slope c_1 shows that π is affine imposing in all intervals with slope c_1 and these slopes are connected, i.e., $\bar{\pi}'(x) = \bar{c}_1$ for all x such that $\pi'(x) = c_1$.

Similarly, we find that π is affine imposing in all intervals with slope c_3 and these slopes are also connected. Furthermore, since π has slope c_3 on the interval $[f, 1]$ and $\theta(f) = 1, \theta(1) = 0$ for $\theta = \pi, \pi_1$, the slope c_3 is fixed for each function to $c_3 = \frac{-1}{1-f}$. Therefore $\bar{\pi}'(x) = 0$ for all x such that $\pi'(x) = c_3$.

Since $\bar{\pi}$ is a continuous piecewise linear function and $\bar{\pi}(0) = 0, \bar{\pi}(x) = \int_0^x \bar{\pi}'(t)dt$, where the finitely many values of t where $\bar{\pi}'$ does not exist are ignored. The following calculations stem from the fact that $\bar{\pi}'(x) = 0$ wherever $\pi'(x) = c_3$, and since π' on $[0, a_2]$ is either of slope c_1 or c_3 . Therefore,

$$\begin{aligned} \bar{\pi}(a_1) - \bar{\pi}(a_0) &= \bar{c}_1 \delta_1^1, \\ \bar{\pi}(a_2) - \bar{\pi}(a_0) &= \bar{c}_1 (\delta_1^1 + \delta_1^2). \end{aligned}$$

Note that $\delta_1^i = (1 - c_1/c_3)\delta^i$. Let $\alpha = (1 - c_1/c_3)$. Let $t_1 = \delta^1$ and $t_2 = \delta^1 + \delta^2$. Then

$$\begin{aligned} \bar{\pi}(a_1) - \bar{\pi}(a_0) &= \bar{c}_1 \alpha t_1, \\ \bar{\pi}(a_2) - \bar{\pi}(a_0) &= \bar{c}_1 \alpha t_2. \end{aligned}$$

Recall that $[A, A_0]$ is the first interval with slope $c_2 = 0$ and let $x \in (x_0 + \Lambda) \cap [A, A_0]$. By Lemma 4.2, we have

$$\begin{aligned}\bar{\pi}(x) - \bar{\pi}(x_0) &= \lambda_1(\bar{\pi}(a_1) - \bar{\pi}(a_0)) + \lambda_2(\bar{\pi}(a_2) - \bar{\pi}(a_0)) \\ &= \lambda_1 \bar{c}_1 \alpha t_1 + \lambda_2 \bar{c}_1 \alpha t_2 \\ &= \alpha \bar{c}_1 (\lambda_1 t_1 + \lambda_2 t_2) \\ &= \alpha \bar{c}_1 (x - x_0).\end{aligned}$$

That is

$$\bar{\pi}(x) = \bar{\pi}(x_0) + \alpha \bar{c}_1 (x - x_0)$$

and therefore, since Λ is dense in \mathbb{R} , $\bar{\pi}(x)$ is affine on a dense set in $[A, A_0]$. Since $\bar{\pi}$ is continuous and $\bar{\pi}$ is affine on all of $[A, A_0]$, the function π is affine imposing in $[A, A_0]$. Since $[A, A_0]$ and $f - [A, A_0]$ are connected via the symmetry reflection, π is also affine imposing in $f - [A, A_0]$ and these intervals must have the same slope.

Since π is continuous with three slopes, we can set up a system of equations on three slopes that is satisfied by π, π^1, π^2 . We will demonstrate this system has a unique solution using just the equations $\pi(f) = 1$, $\pi(1) = 0$, and $\pi(a_0) + \pi(A) = \pi(f - A)$ where $a_0, A, f - A$ are defined above. These equations yield the following system of equations, which is invariant with respect to δ^1, δ^2 :

$$\begin{bmatrix} d_1 & d_2 & d_3 \\ d_1 & d_2 & d_3 + e_3 \\ d_1^1 - d_1^3 & -d_2^1 & d_3^1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (5)$$

Plugging the parameters into the system of equations, we have the matrix equation

$$\begin{bmatrix} 3/5 & 1/10 & 1/10 \\ 3/5 & 1/10 & 3/10 \\ 2/15 - 1/10 & -1/20 & 1/60 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \quad (6)$$

This system has a unique solution, which is $c_1 = 5/2, c_2 = 0, c_3 = -5$. Therefore, $\pi = \pi^1 = \pi^2$ and π is extreme. \square

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